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An Inverse Problem for an Unknown Source in a Heat Equation

J. R. CANNON

Department of Mathematics, University of Texas, Austin, Texas 78712

AND

P. DUCHATEAU

Department of Mathematics, Colorado State University, Fort Collins, Colorado 80523

Submitted by W. F. Ames

1. INTRODUCTION

For T a positive number, suppose $Q_T = \{(x, t): 0 < x, 0 < t < T\}$ and suppose $u(x, t)$ satisfies

$$\partial_t u = \partial_{xx} u + S(u(x, t)) \qquad \text{in} \quad Q_T, \tag{1.1}$$

$$u(x, 0) = 0, \qquad x > 0, \tag{1.2}$$

$$-\partial_x u(0, t) = g(t), \qquad 0 < t < T. \tag{1.3}$$

where the source term $S(u)$ is unknown and is, in fact, to be determined from the "overspecified data,"

$$u(0, t) = f(t), \qquad 0 < t < T. \tag{1.4}$$

That is, the problem (1.1) through (1.3) is completely determined in the sense that if $S(u)$ and $g(t)$ are given then there exists a unique solution $u = u(x, t)$, (provided S and g satisfy certain hypotheses). Then condition (1.4) represents additional information from which we aim to show $S(u)$ can be determined or at least approximated.

It follows, for example, from (1.4) and (1.1) that

$$S(f(t)) = f'(t) - \partial_{xx} u(0, t), \qquad 0 < t < T, \tag{1.5}$$

and hence, if we adopt the notation,

$$\phi(\lambda) = [u(0, t)]^{-1}(\lambda) \tag{1.6}$$

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then (1.1) through (1.3) reduces to

$$\partial_t u = \partial_{xx} u + f'(\phi(u(x, t))) - \partial_{xx} u(0, \phi(u(x, t))) \qquad \text{in} \qquad Q_T \tag{1.7}$$

together with the conditions (1.2), (1.3). We are not prepared at this point to attempt an analysis of Eq. (1.7). Instead we replace (1.7) by the approximate equation

$$\partial_t u(x, t) = \partial_{xx} u + f'(\phi(u(x, t))) - h^{-2}[f(\phi(u)) - 2u(h, \phi(u)) + u(2h, \phi(u))], \tag{1.8}$$

where h denotes a small, positive number. Then as h shrinks toward zero, Eq. (1.8) tends toward Eq. (1.7) and we can hope to show in a future paper that the solution of the problem (1.8), (1.2), (1.3) tends toward the solution of the problem (1.7), (1.2), (1.3).

Note that the first two source terms in Eq. (1.8) are of the form

$$S(u) = F_1(\phi(u)) \tag{1.9}$$

while the second two source terms are of the form

$$S(u) = F_2(u(h, \phi(u))). \tag{1.10}$$

This leads us to consider the following two problems:

$$\begin{aligned} \partial_t u &= \partial_{xx} u + F_1(\phi(u(x, t))) && \text{in} \quad Q_T, \\ u(x, 0) &= 0, && x > 0, \\ -\partial_x u(0, t) &= g(t), && 0 < t < T, \end{aligned} \tag{1.11}$$

and, for $h > 0$,

$$\begin{aligned} \partial_t u &= \partial_{xx} u + F_2(u(h, \phi(u(x, t)))) && \text{in} \quad Q_T, \\ u(x, 0) &= 0, && x > 0, \\ -\partial_x u(0, t) &= g(t), && 0 < t < T, \end{aligned} \tag{1.12}$$

where in each problem we are seeking a function ϕ such that (1.6) holds. In Sections 2 and 3 of this paper we will show for problems (1.11) and (1.12), respectively, that there exists a unique function ϕ for which the corresponding solution $u = u(x, t)$ of the initial-boundary-value problem satisfies (1.6) for T sufficiently small. Then in Section 4 we will apply these results toward finding a solution to (1.8), (1.2), (1.3). Finally, in Section 5 we present the results of some numerical experiments based on Section 4.

The reader will, of course, recognize that we are dealing here with a type of inverse problem. The reader will perhaps also recognize that the approach described here is quite different from the "standard" approach to inverse problems of this sort.

Roughly speaking, the standard approach to this type of inverse problem involves selecting the unknown item, in this case the source function $S(u)$, in such a way that the corresponding unique solution $u(x, t)$ of the (well-posed) problem (1.1), (1.2), (1.3) is, in some sense, optimal with respect to the over-specification (1.4).

This optimization may be effected in a variational setting, (cf. [1]) or in a control theory setting (cf. [2]) or in some context suggested perhaps by the problem itself. In nearly every case, however, when the item being sought is a *function* (as opposed to one or more parameters) these approaches require the analyst to a priori select a form for the function and thereby reduce the problem to one of identifying a finite number of parameters. The present approach is, by contrast, completely form free and allows the unknown source to assume whatever form is consistent with the data and the structure of the problem. This advantage is offset to some degree, however, by the fact that the method is extremely sensitive. In particular, numerical implementation of the scheme requires considerable care and extensive computer resources.

2. Problem 1

Suppose $u(x, t)$ is a function which satisfies, for some $T > 0$,

$$\partial_t u(x, t) = \partial_{xx} u(x, t) + F_1(\phi(u(x, t))) \qquad \text{in} \quad Q_T, \tag{2.1}$$

$$u(x, 0) = 0, \qquad x > 0, \tag{2.2}$$

$$-\partial_x u(0, t) = g(t), \qquad 0 < t < T, \tag{2.3}$$

where we assume

$$g \in \mathbb{C}' \quad \text{with} \quad g(0) > 0 \qquad \text{and} \qquad 0 \leqslant g'(t) \leqslant K, \quad \text{for } t \geqslant 0. \tag{2.4}$$

F_1 is defined and continuous on $[0, \infty)$ and there exist constants $C_1, C_2 > 0$ such that

$$|F_1(x)| \leqslant C_1 \qquad \text{for } x \geqslant 0, \tag{2.5}$$

$$|F_1(x) - F_1(y)| \leqslant C_2 |x - y|, \qquad x, y \geqslant 0.$$

The function ϕ is assumed to belong to the function class S_M to be defined later.

Suppose now that $u_+(x, t)$, $u_-(x, t)$ denote, respectively, the solutions of

$$\partial_t u_\pm(x, t) = \partial_{xx} u_\pm(x, t) \pm C_1 \qquad \text{in} \quad Q_T, \tag{2.6}$$

$$u_\pm(x, 0) = 0, \qquad x > 0, \tag{2.7}$$

$$-\partial_x u_\pm(0, t) = g(t), \qquad 0 < t. \tag{2.8}$$

Then we easily calculate

$$u_{\pm}(x, t) = \int_0^t k(x, t-\tau)\, g(\tau)\, d\tau \pm C_1 \int_0^t \int_0^\infty N(x, \xi; t-\tau)\, d\xi\, d\tau, \tag{2.9}$$

where

$$k(x, t) = \frac{1}{(\pi t)^{1/2}} e^{-x^2/4t}, \qquad x, t > 0, \tag{2.10}$$

and

$$N(x, \xi; t) = \tfrac{1}{2}[k(x-\xi, t) + k(x+\xi, t)]. \tag{2.11}$$

Note that $w_{\pm}(x, t) = \partial_x u_{\pm}(x, t)$ satisfies

$$\begin{aligned} \partial_t w_{\pm}(x, t) &= \partial_{xx} w_{\pm}(x, t) && \text{in} \quad Q_T, \\ w_{\pm}(x, 0) &= 0, && x > 0, \\ w_{\pm}(0, t) &= -g(t), && 0 < t. \end{aligned}$$

A simple maximum principle argument then shows that

$$w_{\pm}(x, t) \equiv \partial_x u_{\pm}(x, t) \leqslant 0 \qquad \text{in} \quad Q_T,$$

from which it follows that

$$u_{\pm}(x, t) \leqslant u_{\pm}(0, t) \qquad \text{for } (x, t) \in Q_T. \tag{2.12}$$

Then since

$$u_{\pm}(0, t) = g(0) \left(\frac{4t}{\pi}\right)^{1/2} + \int_0^t \left(\frac{4\tau}{\pi}\right)^{1/2} g'(t-\tau)\, d\tau \pm C_1 t, \tag{2.13}$$

it follows that for $(x, t) \in Q_T$,

$$-C_1 t \leqslant u_-(x, t) \leqslant u_+(x, t) \leqslant P_+(t), \tag{2.14}$$

where

$$P_+(t) = g(0) \left(\frac{4t}{\pi}\right)^{1/2} + C_1 t + \frac{2K}{3} \left(\frac{4t^3}{\pi}\right)^{1/2}. \tag{2.15}$$

For T a fixed, positive number let

$$\mu_T = P_+(T), \tag{2.16}$$

and for M another fixed, positive constant, we define a class of functions $S(M, T)$ as follows:

DEFINITION 2.1. A function $\phi(u)$, defined for all u, belongs to $S(M, T)$ if

(i) $\phi(u) = 0$ for all $u \leqslant 0$.

(ii) $0 \leqslant \phi(u) \leqslant T$ for all u, $0 \leqslant u \leqslant \mu_T$, and there exists a $\hat{u} \in [0, \mu_T]$ such that $\phi(\hat{u}) = T$. (2.17)

(iii) $0 \leqslant \phi(u_2) - \phi(u_1) \leqslant M(u_2 - u_1)$ for all u_1, u_2 such that $u_1 < u_2$.

Remark. It follows from (2.17iii) that ϕ is nondecreasing and hence $\phi(u) = T$ for all $u \geqslant \hat{u}$, where $\hat{u}$, as in (2.17ii), denotes the first value of u such that $\phi(u) = T$.

Suppose now that $u(x, t)$ satisfies (2.1) through (2.3) for g, F_1 satisfying (2.4), (2.5), respectively, and for ϕ in $S(M, T)$. Then we have the following facts about $u(x, t)$.

LEMMA 2.1. *Under the stated hypotheses,*

$$u_-(x, t) \leqslant u(x, t) \leqslant u_+(x, t), \qquad \forall\, (x, t) \in \bar{Q}_T. \tag{2.18}$$

Proof. Let $z(x, t) = u(x, t) - u_+(x, t)$. Then

$$\begin{aligned} \partial_{xx}z - \partial_t z &= C_1 - F_1(\phi(u(x, t))) \geqslant 0 && \text{in} \quad Q_T, \\ z(x, 0) &= 0, && x > 0, \\ -\partial_x z(0, t) &= 0, && 0 < t < T. \end{aligned}$$

Then it follows from a version of the maximum principle (cf. Theorem 3 of [3, p. 170]) that

$$z(x, t) = u(x, t) - u_+(x, t) \leqslant 0 \qquad \text{in} \quad \bar{Q}_T.$$

Similarly $u_-(x, t) - u(x, t) \leqslant 0$ in $\bar{Q}_T$. This proves the lemma.

LEMMA 2.2. *Under the same hypotheses,*

$$|\,\partial_x u(x, t)| \leqslant g(T) + C_1(4T/\pi)^{1/2} \qquad \textit{for all}\ (x, t) \in \bar{Q}_T. \tag{2.19}$$

Proof. We write

$$u(x, t) = \int_0^t k(x, t - \tau)\, g(\tau)\, d\tau + \int_0^t \int_0^\infty N(x, \xi; t - \tau)\, F_1(\phi(u(\xi, \tau)))\, d\xi\, d\tau,$$

and (2.20)

$$u_x(x, t) = \int_0^t k_x(x, t - \tau)\, g(\tau)\, d\tau + \int_0^t \int_0^\infty N_x(x, \xi; t - \tau)\, F_1(\phi(u(\xi, \tau)))\, d\xi\, d\tau.$$

Then (2.5), (2.10), and (2.11) together imply

$$\sup_{Q_T} |u_x(x,t)| \leqslant g(T) + C_1 \left(\frac{4T}{\pi}\right)^{1/2}.$$

For convenience we will adopt the notation

$$G(T) = g(T) + C_1 \left(\frac{4T}{\pi}\right)^{1/2}, \tag{2.22}$$

and

$$H(T) = \frac{g(0)}{(\pi T)^{1/2}} - MC_2 G(T) \left(\frac{4T}{\pi}\right)^{1/2}, \tag{2.23}$$

where C_2 is the constant appearing in (2.5). Clearly there exists a value $T_M > 0$ such that

$$H(T_M) = 1/M. \tag{2.24}$$

Then if we choose

$$T_* = \min\left[T_M \frac{M}{M+1}, \frac{1}{3MC_2}\right] \tag{2.25}$$

we have the following.

LEMMA 2.3. *Suppose g, F_1 satisfy* (2.4), (2.5), *respectively, and that ϕ belongs to $S(M, T_*)$ for T_* given by* (2.25). *Then $u(x, t)$ satisfies*

$$\min_{[0,T_*]} \partial_t u(0, t) > 1/M. \tag{2.26}$$

Proof. From (2.20) we compute

$$\begin{aligned}
\partial_t u(x,t) &= \int_0^t k_t(x, t-\tau)\, g(\tau)\, d\tau \\
&\quad + F_1(\phi(u(x,t))) + \int_0^t\int_0^\infty N_t(x,\xi;t-\tau) F_1(\phi(u(\xi,\tau)))\, d\xi\, d\tau \\
&= \int_0^t k(x,\tau)\, g'(t-\tau)\, d\tau + k(x,t)\, g(0) \\
&\quad + F_1(\phi(u(x,t))) + \int_0^t\int_0^\infty N_{\xi\xi}(x,\xi;t-\tau) F_1(\phi(u(\xi,\tau)))\, d\xi\, d\tau.
\end{aligned}$$

Now write

$$\begin{aligned}
&\int_0^t\int_0^\infty N_{\xi\xi}(x,\xi;t-\tau) F_1(\phi(u(\xi,\tau)))\, d\xi\, d\tau \\
&\quad = \lim_{h\downarrow 0} \int_0^t\int_0^\infty [N_\xi(x,\xi+h,t-\tau) - N_\xi(x,\xi;t-\tau)]\, F_1(\phi(u(\xi,\tau)))\, h^{-1}\, d\xi\, d\tau.
\end{aligned}$$

But

$$\int_0^\infty N_\xi(x, \xi + h, t - \tau) F_1(\phi(u(\xi, \tau)))\, d\xi$$
$$= \int_h^\infty N_\xi(x, \xi, t - \tau) F_1(\phi(u(\xi - h, \tau)))\, d\xi$$

and hence

$$\int_0^\infty \{N_\xi(x, \xi + h, t - \tau) - N_\xi(x, \xi, t - \tau)\} F_1(\phi(u(\xi, \tau)))\, h^{-1}\, d\xi$$
$$= \int_h^\infty N_\xi(x, \xi, t - \tau) \{F_1(\phi(u(\xi - h, \tau))) - F_1(\phi(u(\xi, \tau)))\}\, h^{-1}\, d\xi$$
$$- \int_0^h N_\xi(x, \xi, t - \tau) F_1(\phi(u(\xi, \tau)))\, h^{-1}\, d\xi.$$

We observe that

$$\lim_{h \downarrow 0} | h^{-1}\{F_1(\phi(u(\xi - h, \tau))) - F_1(\phi(u(\xi, \tau)))\}|$$
$$\leqslant C_2 M \lim_{h \downarrow 0} | h^{-1}(u(\xi - h, \tau) - u(\xi, \tau))|$$
$$\leqslant C_2 M \sup_{Q_T^*} | \partial_x u(x, t)| ,$$

and, in addition,

$$\lim_{h \downarrow 0} \int_0^h N_\xi(0, \xi; t - \tau) F_1(\phi(u(\xi, \tau)))\, h^{-1}\, d\xi = 0$$

since

$$N_\xi(0, \xi; t - \tau) \leqslant \text{Const. } h, \qquad \text{for } 0 \leqslant \xi \leqslant h.$$

Combining all these results, we obtain the following estimate, valid for all t, $0 \leqslant t \leqslant T_*$,

$$\partial_t u(0, t) \geqslant \frac{g(0)}{(\pi t)^{1/2}} - MC_2 \sup_{Q_{T_*}} | \partial_x u | \int_0^{T_*} \int_0^\infty N_\xi(0, \xi, t - \tau)\, d\xi\, d\tau$$

or, in the view of the previous lemma,

$$\partial_t u(0, t) \geqslant \frac{g(0)}{(\pi t)^{1/2}} - MC_2 G(T^*) \left(\frac{4T_*}{\pi}\right)^{1/2}, \qquad 0 \leqslant t \leqslant T_* .$$

Then, using (2.23), (2.24), (2.25) we have

$$\min_{[0, T_*]} \partial_t u(0, t) > 1/M,$$

proving the lemma.

Suppose now that $u(x, t)$ satisfies (2.1) through (2.3) for g, F_1 satisfying (2.4), (2.5) and for ϕ in $S(M, T_*)$. We are going to denote $S(M, T_*)$ by S_M, since T_* depends on M. Then if

$$f(t) = u(0, t), \qquad 0 \leqslant t \leqslant T_*,$$

it follows from (2.14) and (2.18) that

$$0 \leqslant f(t) \leqslant \mu_{T_*}, \qquad 0 \leqslant t \leqslant T_*, \tag{2.27}$$

and, in addition, (2.26) implies

$$f'(t) > 1/M, \qquad 0 \leqslant t \leqslant T_*. \tag{2.28}$$

Then the inverse function f^{-1} is well defined for $0 \leqslant t \leqslant T_*$ and if we denote this inverse by ψ, we have

$$\psi(f(t)) = t \qquad \text{for } 0 \leqslant t \leqslant T_*, \tag{2.29}$$

and

$$0 \leqslant \psi(u) \leqslant T_* \qquad \text{for } 0 \leqslant u \leqslant \mu_{T_*}. \tag{2.30}$$

More precisely, we have

$$0 \leqslant \psi(u) \leqslant T_* \qquad \text{for } 0 \leqslant u \leqslant \hat{u},$$

where

$$\hat{u} = f(T_*) \leqslant \mu_{T_*},$$

and we then extend the function as follows:

$$\begin{aligned} \psi(u) &= 0 && \text{for } u \leqslant 0, \\ \psi(u) &= T_* && \text{for } u \geqslant \hat{u}. \end{aligned} \tag{2.31}$$

Now (2.28) implies that

$$0 < t_2 - t_1 < M(f(t_2) - f(t_1)),$$

for $0 \leqslant t_1 < t_2 \leqslant T_*$. Then (2.29) implies

$$0 < \psi(f(t_2)) - \psi(f(t_1)) < M(f(t_2)) - f(t_1)),$$

or, letting $u_i = f(t_i)$, $i = 1, 2$,

$$0 < \psi(u_2) - \psi(u_1) \leqslant M(u_2 - u_1), \tag{2.32}$$

for $0 \leqslant u_1 < u_2 \leqslant \hat{u}$. In view of the definition (2.31) extending the function ψ, it is evident that (2.32) holds for all $u_1 \leqslant u_2$ and consequently, ψ must belong to S_M.

LEMMA 2.4. *Suppose that g, F_1 satisfy* (2.4), (2.5), *respectively. Then for each ϕ in S_M there exists a unique solution, $u = u(x, t)$, for* (2.1) *through* (2.3) *and the solution u defines a unique function ψ in S_M via* (2.29), (2.31). *If we denote this correspondence between ϕ and ψ by writing $\psi = \not{f}(\phi)$, then $\not{f}$ is a contraction mapping of S_M into itself.*

Proof. It is well known that under the hypotheses (2.4) on g and (2.5) on F_1 and for ϕ in S_M there will always exist a unique solution $u = u(x, t)$ for (2.1) through (2.3) (cf. [4]). What has to be proved is that this solution $u(x, t)$ has the property that $u(0, t)$ is invertible as a function of t and that the inverse ψ belongs to S_M. Finally, we must show that the mapping $\not{f}$, which assigns to each ϕ in S_M the image ψ in S_M, is a contraction.

We have already seen that ψ defined by (2.29), (2.31) belongs to S_M. In order to show that $\not{f}$ is a contraction let ϕ_1, ϕ_2 denote two elements of S_M and let u_1, u_2 denote the corresponding solutions of (2.1), through (2.3) in $\hat{D}_{T_*}$. Then

$$u_1(0, t) - u_2(0, t) = \int_0^t \int_0^\infty N(0, \xi; t - \tau)\,[F_1(\phi_1(u_1(\xi, \tau))) - F_1(\phi_2(u_2(\xi, \tau)))]\, d\xi\, d\tau$$

and

$$\begin{aligned} |f_1(t) - f_2(t)| &\leqslant C_2 \int_0^t \int_0^\infty N(0, \xi; t - \tau)\, |\phi_1(u_1(\xi, \tau)) - \phi_2(u_2(\xi, \tau))|\, d\xi\, d\tau \\ &\leqslant C_2 \int_0^t \int_0^\infty N(0, \xi; t - \tau)\, |\phi_1(u_1(\xi, \tau)) - \phi_2(u_1(\xi, \tau))|\, d\xi\, d\tau \\ &\quad + C_2 \int_0^t \int_0^\infty N(0, \xi; t - \tau)\, |\phi_2(u_1(\xi, \tau)) - \phi_2(u_2(\xi, \tau))|\, d\xi\, d\tau. \end{aligned}$$

Finally, since

$$\int_0^t \int_0^\infty N(0, \xi; t - \tau)\, d\xi\, d\tau = t$$

we have

$$\sup_{[0,T_*]} |f_1(t) - f_2(t)| \leqslant C_2 T_* \sup_{[0,\mu_{T_*}]} |\phi_1(u) - u_2(u)| + M T_* C_2 \sup_{[0,T_*]} |f_1(t) - f_2(t)|$$

and

$$\sup_{[0,T_*]} |f_1 - f_2| \leqslant \frac{C_2 T_*}{1 - M T_* C_2} \sup_{[0,\mu_{T_*}]} |\phi_1(u) - \phi_2(u)|\,.$$

Next observe that if $0 \leqslant t_1 < t_2 \leqslant T_*$, then

$$\frac{f_1(t_2) - f_2(t_2)}{\psi_2(u) - \psi_1(u)} = f_1'(\tau) > \frac{1}{M} \qquad \text{for some } \tau, \quad t_1 \leqslant \tau \leqslant t_2,$$

where

$$f_1(t_1) = u = f_2(t_2).$$

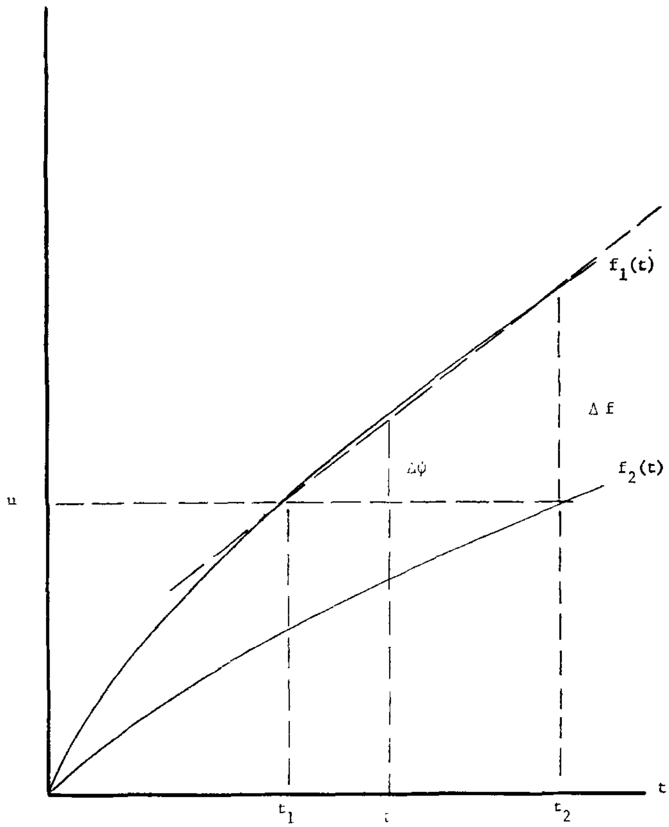

FIGURE 1

It follows that

$$\sup_{[0,T_*]} |f_1 - f_2| > \frac{1}{M} \sup_{[0,u_{T_*}]} |\psi_1(u) - \psi_2(u)|,$$

and hence

$$\sup_{[0,U_*]} |\psi_1(u) - \psi_2(u)| \leqslant \frac{MC_2T_*}{1 - MC_2T_*} \sup_{[0,u_{T_*}]} |\phi_1(u) - \phi_2(u)|.$$

Then (2.25) implies that

$$\| \psi_1 - \psi_2 \| \leqslant \tfrac{1}{2} \| \phi_1 - \phi_2 \|,$$

and $\not{f}$ is a contraction from the continuous functions into the space of continuous functions. It remains to show that the unique fixed point, which perforce exists, is in fact an element of S_M.

Conditions (i) and (ii) of (2.17) hold uniformly for each iterate in the mapping sequence and hence they must hold for ψ_∞. Moreover, if $\{\psi_n\}$ is a sequence of iterates converging to ψ_∞ then for each n,

$$M(u_1 - u_2) > \psi_n(u_1) - \psi_n(u_2) \geqslant 0.$$

If we write

$$\psi_n(u_1) - \psi_n(u_2) = \psi_n(u_1) - \psi_\infty(u_1) + \psi_\infty(u_1) - \psi_\infty(u_2) + \psi_\infty(u_2) - \psi_n(u_2),$$

then clearly

$$M(u_1 - u_2) > \psi_\infty(u_1) - \psi_\infty(u_2) - | \psi_n(u_1) - \psi_\infty(u_1)| - | \psi_n(u_2) - \psi_\infty(u_2)|,$$

and since the left side of this last inequality is independent of n, we may let n go to infinity on the right so as to obtain

$$M(u_1 - u_2) > \psi_\infty(u_1) - \psi_\infty(u_2) \geqslant 0.$$

Then ψ_∞ is in S_M and the lemma is proved. We now have the main result of this section.

THEOREM 1. *There exists a unique element ψ_∞ in S_M such that*

$$\begin{aligned} \partial_t u(x, t) &= \partial_{xx} u(x, t) + F_1(\psi_\infty(u(x, t))) &&\text{in} \quad Q_{T_*}, \\ u(x, 0) &= 0, && x > 0, \\ -\partial_x u(0, t) &= g(t), && 0 < t < T_*, \end{aligned}$$

has a unique solution $u_\infty(x, t)$ and, moreover,

$$\psi_\infty(u_\infty(0, t)) = t \qquad \text{for } 0 \leqslant t \leqslant T_*.$$

Of course, we continue to suppose here that g, F_1 satisfy (2.4), (2.5).

Proof. The theorem follows directly from the previous lemma.

3. Problem 2

We consider now the second of the two problems motivated by the problem of identifying an unknown source in the heat equation from data measured at the boundary.

That is, for some $T > 0$, we suppose that $u(x, t)$ satisfies

$$\partial_t u(x, t) = \partial_{xx} u(x, t) + F_2(u(h, \phi(u(x, t)))) \quad \text{in} \quad Q_T, \tag{3.1}$$

$$u(x, 0) = 0, \quad x > 0, \tag{3.2}$$

$$-\partial_x u(0, t) = g(t), \quad 0 < t < T. \tag{3.3}$$

We continue to suppose that $g(t)$ satisfies (2.4) and that F_2 satisfies (2.5). In addition, we suppose that ϕ belongs to the class $S(M, T)$ for positive constants M, T.

It follows then that Lemmas 2.1 and 2.2 apply to the solution $u(x, t)$ of (3.1) through (3.3). In addition, we have

Lemma 3.1. *For g, F_2 as described, for $h > 0$ and for $T > 0$ sufficiently small,*

$$\sup_{[0,T]} | \partial_t u(h, t)| \leqslant (4/eh)(g(0) + KT) + 2C_1. \tag{3.4}$$

Proof. For $h > 0$, we compute from (2.20),

$$\partial_t u(h, t) = k(h, t) g(0) + \int_0^t k(h, \tau) g'(t - \tau) \, d\tau + F_2(u(h, \phi(u(h, t))))$$
$$+ \int_0^t \int_0^\infty N_{\xi\xi}(h, \xi; t - \tau) F_2(u(h, \phi(u(\xi, \tau)))) \, d\xi \, d\tau.$$

Now

$$k(h, t) = \frac{e^{-h^2/2t}}{(\pi t)^{1/2}} \leqslant \frac{2}{eh} \qquad \text{for all } t \geqslant 0,$$

and hence,

$$\sup_{[0,T]} | \partial_t u(h, t)| \leqslant (2/eh)(g(0) + KT) + C_1 + \sup_{[0,T]} | w(h, t)|, \tag{3.5}$$

where

$$w(h, t) = \int_0^t \int_0^\infty N_{\xi\xi}(h, \xi; t - \tau) F_2(u(h, \phi(u(\xi, \tau)))) \, d\xi \, d\tau. \tag{3.6}$$

In order to simplify the proofs of this and the next lemma somewhat, we are going to proceed now as if F_2 and ϕ are differentiable functions instead of just Lipschitz continuous. The proofs in the case of the Lipschitz hypothesis are quite similar to the proof of Lemma 2.3. Under the hypothesis of differentiability, we can write

$$w(h,t) = \int_0^t N_\xi(h,\xi;t-\tau)\,F_2(u(h,\phi(u(\xi,\tau))))\Big|_{\xi=0}^{\xi=\infty} d\tau$$
$$- \int_0^t\int_0^\infty N_\xi(h,\xi;t-\tau)\,F'(u)\,\partial_t u(h,\phi(u))\,\phi'(u)\,\partial_\xi u(\xi,\tau)\,d\xi,\,d\tau. \tag{3.7}$$

But

$$\lim_{\xi\downarrow 0} N_\xi(h,\xi;t-\tau) = 0, \tag{3.8}$$

$$\lim_{\xi\downarrow\infty} N_\xi(h,\xi;t-\tau) = 0, \tag{3.9}$$

and

$$\int_0^t\int_0^\infty N_\xi(h,\xi;t-\tau)\,d\xi\,d\tau \leqslant \left(\frac{4t}{\pi}\right)^{1/2}, \qquad t > 0. \tag{3.10}$$

We are going to suppose that the derivatives F' and ϕ' are bounded in absolute value by constants C_2 and M, respectively. If we let

$$B_2(T) = \max_{[0,T]} |\,\partial_t u(h,t)| \tag{3.11}$$

and

$$B_3(T) = \max_{[0,T]} |\,w(h,t)| \tag{3.12}$$

then we can rewrite (3.5) as

$$B_2(T) \leqslant (2/eh)\,(g(0) + KT) + C_1 + B_3(T), \tag{3.13}$$

and we proceed now to estimate $B_3(T)$. From (3.7) through (3.10) and (2.19), (2.22) we derive

$$B_3(T) \leqslant MC_2G(T)\left(\frac{4T}{\pi}\right)^{1/2}\left\{\frac{2}{eh}\,(g(0) + KT) + C_1 + B_3(T)\right\}$$

and

$$B_3(T) \leqslant \frac{MC_2G(T)\,(4T/\pi)^{1/2}\,(2/eh)\,(g(0) + KT) + C_1}{1 + MC_2G(T)\,(4T/\pi)^{1/2}} \tag{3.14}$$

It is evident that we may choose a value $T_0 > 0$ such that

$$MC_2G(T_0)\,(4T_0/\pi)^{1/2} = \tfrac{1}{2} \tag{3.15}$$

and then for $T \leqslant T_0$ we have

$$B_3(T) \leqslant (2/eh)\,(g(0) + KT) + C_1\,. \tag{3.16}$$

Then the lemma follows from (3.16) and (3.13).

Now we have

LEMMA 3.2. *For g, F_2 satisfying* (2.4), (2.5), *respectively, and ϕ in $S(M, T)$, there exists a T_1, $0 < T_1 < T$, such that*

$$\min_{[0,T_1]} \partial_t u(0, t) > 1/M. \tag{3.17}$$

Proof. We write

$$\partial_t u(0, t) = k(0, t)\, g(0) + \int_0^t k(0, \tau)\, g'(t - \tau)\, d\tau + F_2(u(h, \phi(u(0, t))))$$
$$+ \int_0^t \int_0^\infty N_{\xi\xi}(0, \xi; t - \tau)\, F_2(u(h, \phi(u(\xi, \tau))))\, d\xi\, d\tau$$

and then

$$\partial_t u(0, t)$$
$$\geqslant \frac{g(0)}{(\pi t)^{1/2}} - \int_0^t \int_0^\infty | N_\xi(0, \xi; t - \tau)\, F_2'(u)\, \partial_t u(h, \phi(u))\, \phi'(u)\, \partial_\xi u(\xi, \tau)|\, d\xi\, d\tau,$$

where we have used again the modified hypotheses on F_2 and ϕ. It follows now from the previous lemma, (2.19), (2.22), and (3.10) that

$$\partial_t u(0, t) \geqslant \frac{g(0)}{(\pi t)^{1/2}} - MC_2 G(T) \left(\frac{4T}{\pi}\right)^{1/2} \left\{\frac{4}{eh}(g(0) + KT) + 2C_1\right\} \tag{3.18}$$

for any $T \leqslant T_0$ and all t, $0 < t < T - T_0$. Let

$$H(T) = \frac{g(0)}{(\pi T)^{1/2}} - MC_2 G(T) \left(\frac{4T}{\pi}\right)^{1/2} \left\{\frac{4}{eh}(g(0) + KT) + 2C_1\right\} \tag{3.19}$$

and observe that there exists some $T_M > 0$ such that $H(T_M) = 1/M$. Thus, if we choose

$$T_1 = \min\left(T_0\,,\, T_M \frac{M}{M + 1}\right) \tag{3.20}$$

for T_0 given by (3.15), then the lemma follows.

Remark. We emphasize that the Lemmas 3.1 and 3.2 require only the hypothesis of Lipschitz continuity on F_2 and ϕ as expressed in (2.5) and (2.17). We have used the hypothesis of differentiability for these functions only for convenience in the proofs of the lemmas.

We proceed now as in the previous section. We write S_M for $S(M, T_1)$ since T_1 depends on M, and we suppose that $u(x, t)$ satisfies (3.1) through (3.3) for g, F_2 satisfying (2.4), (2.5) and for ϕ in S_M. Then let

$$f(t) = u(0, t) \tag{3.21}$$

and let $\psi(u)$ be defined as in (2.29) and (2.31). Then it follows, just as before, that ψ belongs to S_M. We have, in fact,

LEMMA 3.3. *Suppose that g, F_2 satisfy* (2.4), (2.5), *respectively. Then for each ϕ in S_M there exists a unique solution $u = u(x, t)$ for* (3.1) *through* (3.3), *and this solution u defines a unique function ψ in S_M via* (3.21), (2.29), *and* (2.31). *If we write $\psi = \mathscr{f}(\phi)$, then $\mathscr{f}$ is a contraction mapping of S_M into itself.*

Proof. The proof of this result differs from the proof of Lemma 2.4 only in showing that $\mathscr{f}$ is a contraction. We let ϕ_1, ϕ_2 denote two elements of S_M and let $u_1(x, t)$ and $u_2(x, t)$ denote the corresponding solutions of (3.1) through (3.3) in Q_{T_1}. Then

$$\begin{aligned} &u_1(x, t) - u_2(x, t) \\ &\quad = \int_0^t \int_0^\infty N(x, \xi; t-\tau)\,[F_2(u_1(h, \phi_1(\xi, \tau))) - F_2(u_2(h, \phi_2(u_2(\xi, \tau)))]\,d\xi\,d\tau \end{aligned}$$

for all $(x, t) \in Q_{T_1}$ and we can write

$$\begin{aligned} u_1(x, t) &- u_2(x, t) \\ &= \int_0^t \int_0^\infty N(x, \xi; t-\tau)\,[F_2(u_1(h, \phi_1(u_1))) - F_2(u_1(h, \phi_1(u_2))] \\ &\quad + \int_0^t \int_0^\infty N(x, \xi; t-\tau)\,[F_2(u_1(h, \phi_1(u_2))) - F_2(u_2(h, \phi_1(u_2)))]\,d\xi\,d\tau \\ &\quad + \int_0^t \int_0^\infty N(x, \xi; t-\tau)\,[F_2(u_2(h, \phi_1(u_2))) - F_2(u_2(h, \phi_2(u_2)))]\,d\xi\,d\tau. \end{aligned}$$

Now

$$\int_0^t \int_0^\infty N(x, \xi; t-\tau)\,d\xi\,d\tau = t \qquad \text{for } t > 0, \tag{3.22}$$

and hence

$$\begin{aligned} \sup_{Q_{T_1}} | u_1 - u_2 | \leqslant C_2 B_2(T_1)\, M T_1 &\sup_{Q_{T_1}} | u_1 - u_2 | \\ &+ C_2 T_1 \sup_{Q_{T_1}} | u_1 - u_2 | + C_2 B_2(T_1)\, T_1 \sup_{[0, \mu T_1]} | \phi_1 - \phi_2 |, \end{aligned} \tag{3.23}$$

in view of (2.5), (2.16), and (3.11). It follows now from (3.23) that

$$\sup_{Q_{T_1}} | u_1 - u_2 | \leqslant \frac{C_2 B_2(T_1)\, T_1}{1 - C_2 T_1(1 + MB_2(T_1))} \sup_{[0,\mu_{T_1}]} | \phi_1 - \phi_2 |$$

and in particular,

$$\sup_{[0,T_1]} | f_1(t) - f_2(t) | \leqslant \frac{C_2 B_2(T_1)\, T_1}{1 - C_2 T_1(1 + MB_2(T_1))} \sup_{[0,\mu_{T_1}]} | \phi_1 - \phi_2 | . \tag{3.24}$$

We can show now, just as we did in the proof of Lemma 2.4, that

$$\sup_{[0,T_1]} | f_1 - f_2 | \geqslant (1/M) \sup_{[0,\mu_{T_1}]} | \psi_1 - \psi_2 | , \tag{3.25}$$

where

$$\mu_{T_1} = P_+(T_1).$$

Then it follows from (3.24), (3.25) that

$$\sup_{[0,\mu_{T_1}]} | \psi_1 - \psi_2 | \leqslant \frac{C_2 M B_2(T_1)\, T_1}{1 - C_2 T_1(1 + MB_2(T_1))} \sup_{[0,\mu_{T_1}]} | \phi_1 - \phi_2 | . \tag{3.26}$$

If we let $T_2 > 0$ be such that

$$C_2 T_2(1 + MB_2(T_2)) = \tfrac{1}{2}$$

and choose $T_1 > 0$ to be the smaller of its current value and the value T_2, then it follows from (3.26) that $\not f$ is a contraction of S_M into itself.

The remainder of the proof is identical to the proof of Lemma 2.4.

Now we have

THEOREM 2. *There exists a unique element ψ_∞ in S_M such that*

$$\begin{aligned} \partial_t u(x, t) &= \partial_{xx} u(x, t) + F_2(u(h, \psi_\infty(u(x, t)))) && \text{in} \quad Q_{T_1}, \\ u(x, 0) &= 0, && x > 0, \\ -\partial_x u(0, t) &= g(t), && 0 < t < T_1, \end{aligned}$$

has a unique solution $u_\infty(x, t)$ satisfying

$$\psi_\infty(u_\infty(0, t)) = t \qquad \textit{for } 0 \leqslant t \leqslant T_1 .$$

We have supposed here that F_2, g satisfy (2.5), (2.4), *respectively.*

The proof follows directly from the lemma.

4. Identification of an Unknown Source

We return now to the problem of identifying an unknown source term $S(u)$ in the heat equation from data measured at the boundary. For reasons cited in the Introduction we are content to consider the approximate problem (1.8), (1.2), (1.3), in conjunction with the overspecification (1.4).

We continue to make use of the definition (1.6) for ϕ and we suppose that the function in (1.4) satisfies

$$f \in C^1[0, \infty) \qquad \text{with} \qquad f(0) = 0. \tag{4.1}$$

In addition, f and f' satisfy (2.5).

We remark that since we are dealing here with an overspecified problem, we must be careful that any hypotheses we place on f are consistent with the other conditions of the problem. It is fairly evident that the condition (4.1) on f does not conflict with any of the conditions (1.8), (1.2), (1.3), (1.4), or (2.4) although the constants C_1 and C_2 in (2.5) which bear on f and f' will depend in some way on the constant K in (2.4).

If we define, for $h > 0$,

$$F_1(w) = f'(w) - h^{-2}f(w), \qquad 0 \leqslant w < \infty, \tag{4.2}$$

then F_1 satisfies (2.5) if f satisfies (4.1).

We also define, for h, $T > 0$,

$$\begin{aligned} F_2(w) &= 0 && \text{for } w < 0, \\ &= h^{-2}w && \text{for } 0 \leqslant w - T, \\ &= h^{-2}T && \text{for } w > T. \end{aligned} \tag{4.3}$$

Then F_2 satisfies (2.5) as well.

We may now rewrite (1.8) as

$$\begin{aligned} &\partial_t u(x, t) - \partial_{xx} u(x, t) \\ &\quad = F_1(\phi(u(x, t))) + 2F_2(u(h, \phi(u(x, t)))) - F_2(u(2h, \phi(u(x, t)))) \qquad \text{in} \quad Q_T \end{aligned} \tag{4.4}$$

and the following theorem is the direct result of Theorems 1 and 2 of the previous sections.

THEOREM 3. *Suppose g satisfies* (2.4) *and f satisfies* (4.1), *and that for* $h > 0$, $T > 0$ *fixed*, F_1 *and* F_2 *are given by* (4.1) *and* (4.3), *respectively. Then there exists a* $T_* > 0$ *and a unique* ϕ_* *in* S_M *such that*

$$\begin{aligned} \partial_t u(x, t) = \partial_{xx} u(x, t) + F_1(\phi_*(u(x, t))) + 2F_2(u(h, \phi_*(u(x, t)))) \\ - F_2(u(2h, \phi_*(u(x, t)))), \qquad \textit{in} \qquad Q_{T*}, \end{aligned} \tag{4.5}$$

$$u(x, 0) = 0, \qquad x > 0, \tag{4.6}$$

$$-\partial_x u(0, t) = g(t), \qquad 0 < t < T_*, \tag{4.7}$$

has a unique solution $u_*(x, t)$ and moreover,

$$\phi_*(u_*(0, t)) = t \qquad \text{for } 0 < t < T_*. \tag{4.8}$$

Now (1.5) suggests that we may approximate the unknown source $S(u)$ appearing in (1.1) by the expression

$$S_*(u) = f'(\phi_*(u)) - h^{-2}[f(\phi_*(u)) - 2u_*(h, \phi_*(u)) + u_*(2h, \phi_*(u))]. \tag{4.9}$$

In the next section we discuss some numerical experiments in which the solution $u_*(x, t)$, ϕ_* *alluded to in Theorem* 3 *is constructed numerically and the approximation $S_*(u)$ is computed.*

5. Numerical Experiments

The numerical experiments used to illustrate the results of Section 4 were conducted in two phases.

In the first phase we simulated the collection of the data $f(t)$ and $g(t)$. More precisely, we solved numerically the following initial-boundary-value problem,

$$\partial_t u(x, t) = \partial_{xx} u(x, t) + S(u(x, t)) \qquad \text{in} \quad Q_T, \tag{5.1}$$

$$u(x, 0) = 0, \qquad 0 < x < 1, \tag{5.2}$$

$$u(0, t) = f(t), \qquad 0 < t < T, \tag{5.3}$$

$$u(1, t) = 0, \qquad 0 < t < T, \tag{5.4}$$

for a ***known*** source $S(u)$ and a given data function $f(t)$. Note that the problem has been restricted to a finite interval for convenience in solving the numerical problem.

Using the numerical solution of (5.1) through (5.4), we *calculated*

$$g(t) = \partial_x u(0, t), \qquad 0 < t < T. \tag{5.5}$$

We point out that (5.1) through (5.4) were solved by means of a fully implicit predictor–corrector scheme and $g(t)$ was computed from a second-order correct (three-point) approximation for the derivative $\partial_x u(0, t)$. Having completed the simulation of the collection of data, we then proceeded to the second phase of the numerical experiment in which we sought to reconstruct the source function $S(u)$ from the "data," f and g.

This phase of the numerical experiment also involved a prediction and a correction step for a fully implicit approximation to Eq. (1.8). To describe the predictor step, let us suppose that the solution $u(x, t)$ has been (approximately) computed at each of the node points x_i, $i = 1,..., N$, for every time level $t = t_1, t_2,...,$ up to t_n. Then $u_i^{n+1} = u(x_i, t_{n+1})$, satisfies

$$\begin{aligned} \Delta_t u_i^{n+1} &= \Delta_x^2 u_i^{n+1} + \hat{S}(u_i^n), \qquad i = 1,..., N, \\ u_i^n &= \text{known}, \\ \Delta_x u_1^{n+1} &= g^{n+1} \qquad \text{and} \qquad u_N^{n+1} = 0. \end{aligned} \tag{5.6}$$

Here we let

$$F_k = u_1^k, \quad k = 1, 2,..., n, \qquad \text{and} \qquad F_0 = 0. \tag{5.7}$$

Then if

$$F_{k-1} \leqslant u_i^n \leqslant F_k \qquad \text{for some } k, \quad 1 \leqslant k \leqslant n, \tag{5.8}$$

we can compute

$$\hat{S}_k(u_i^n) = S_k + \frac{\Delta S_k}{\Delta F_k}(u_i^n - F_k), \tag{5.9}$$

where

$$\begin{aligned} S_k &= 2f'(t_k) - \Delta_x^2 u_1^k \qquad \text{(known)}, \\ \Delta S_k &= S_k - S_{k-1}, \\ \Delta F_k &= F_k - F_{k-1}. \end{aligned} \tag{5.10}$$

In this way the predictor solution u_i^{n+1} at the time level t_{n+1} is completely determined. Having once computed it, we then set

$$F_{n+1} = u_1^{n+1} \tag{5.11}$$

and

$$S_{n+1} = 2f'(t_{n+1}) - \Delta_x^2 u_1^{n+1}. \tag{5.12}$$

This allows us to go now to the corrector step of this phase of the numerical experiment, where we solve

$$\begin{aligned} \Delta_t v_i^{n+1} &= \Delta_x^2 v_i^{n+1} + \hat{S}(u_i^{n+1}), \qquad i = 1,..., N, \\ v_i^n &= u_i^n, \\ \Delta_x u_1^{n+1} &= g^{n+1} \qquad \text{and} \qquad v_N^{n+1} = 0. \end{aligned} \tag{5.13}$$

Having (5.7) and (5.11) together with (5.10) and (5.12) allows us to evaluate $\hat{S}(u_i^{n+1})$ from (5.9) for k running now up to $n+1$. The corrector solution v_i^{n+1} is then completely determined by (5.13) and after calculating it we correct (5.11), (5.12) by letting

$$\begin{aligned} F_{n+1} &= v_1^{n+1}, \\ S_{n+1} &= 2f'(t_{n+1}) - \Delta_x{}^2 v_1^{n+1}. \end{aligned} \tag{5.14}$$

We should point out that the second difference expressions $\Delta_x{}^2 u_1{}^k$ and $\Delta_x{}^2 v_1{}^k$ required for computing S_k are second-order-correct (i.e., five-point) approximations to the derivative $\partial_{xx}u(0, t)$. This order of approximation, together with the mesh size

$$\Delta x = h = 10^{-3}, \qquad \Delta t = 10^{-4}, \tag{5.15}$$

were necessary in order to obtain three significant digits in computing $S(u)$. Using a first-order-correct (three-point) approximation for $\Delta_x{}^2 u_1{}^k$ together with mesh size

$$\Delta x = 10^{-2}, \qquad \Delta t = 10^{-3}, \tag{5.16}$$

we were able to obtain only one significant digit in calculating $S(u)$.

One further word of clarification regarding the difference approximations $\Delta_x u_1^{n+1}$ and $\Delta_x{}^2 u_1^{n+1}$ in (5.6) and (5.10), respectively. Since we are approximating derivatives at the boundary here, the difference expressions must be one-sided as opposed to centered differences. Thus, in order to obtain second-order accuracy, it is necessary to use three-five-point difference expressions, respectively. In particular, we used

$$\Delta_x u_1^{n+1} = \frac{4u_2^{n+1} - 3u_1^{n+1} - u_3^{n+1}}{2\Delta x}, \tag{5.17}$$

and

$$\Delta_x{}^2 u_1^{n+1} = \frac{10u_3^{n+1} - u_5^{n+1} - 16u_2^{n+1} + 7u_1^{n+1}}{4\Delta x^2}. \tag{5.18}$$

We carried out these numerical experiments for three different choices of the source $S(u)$. We used

$$\begin{aligned} S(u) &\equiv 0 && \text{with} \quad f(t) = t^2, \\ S(u) &\equiv -1 && \text{with} \quad f(t) = t^2, \\ S(u) &= -u^2 && \text{with} \quad f(t) = 100t^2. \end{aligned} \tag{5.19}$$

In each case we were able to reconstruct the source $S(u)$, with three significant digits of accuracy. In view of the very fine mesh required, the calculations were rather expensive in terms of computer time. For this reason, the amount of numerical experimentation was not as extensive as we might have liked. These results do, however, indicate that it is at least feasible to construct an unknown source $S(u)$ from overspecified data measured on the boundary by this technique.

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